'समानो मन्त्रः समितिः समानी'

UNIVERSITY OF NORTH BENGAL

B.Sc. Honours 4th Semester Examination, 2023

CC10-MATHEMATICS

METRIC SPACE AND COMPLEX THEORY

(REVISED SYLLABUS 2023 / OLD SYLLABUS 2018)

Time Allotted: 2 Hours | Full Marks: 60

The figures in the margin indicate full marks.

GROUP-A

1. Answer any ***four*** questions from the following: $3\times4 = 12$

(a) Show that the function $f : X \to Y$ is uniformly continuous, where X is discrete metric space and Y is any metric space.

(b) Give an example with proper justification of a set which is bounded but not totally bounded.

(c) Let X be a set and $|X| \geq 2$ with the discrete metric. Show that X is not connected.

(d) If f be an analytic function on a region $G(\subset \mathbb{C})$ such that $\operatorname{Im} f = 0$, then show that f is constant.

(e) Show that every totally bounded metric space is separable.

(f) Find the Laurent series expansion of the function $\dfrac{7z-2}{z(z-2)(z+1)}$ in the domains $|z| > 2$ and $1 < |z| < 2$ respectively. $1\frac{1}{2}+1\frac{1}{2}$

GROUP-B

Answer any *four* questions from the following $6\times4 = 24$

2. Show that the map $f : [0, 1] \to [0, 1]$ given by $f(x) = x - \dfrac{x^2}{2}$, is a weak contraction map but not contraction map. Also find its fixed points if exists. 6

3. Establish Cauchy-Riemann equations in the polar form for a function $f(z)$. 6

4. For any non-empty A of a metric space (X, d), show that the function $f : X \longrightarrow \mathbb{R}$ given by $f(x) = d(x, A)$; $x \in X$, is uniformly continuous. 6

5. State and prove the sufficient conditions for differentiability of a complex valued function $f(z)$ of a complex variable. 6

6. Let $f(z)=u(x,\ y)+iv(x,\ y)$ be an analytic function in a region G. Verify whether the functions $\overline{f(z)}$, $f(\bar{z})$, $\overline{f(\bar{z})}$ are analytic or not in G. 2+2+2

7. (a) Let $f(z)=\dfrac{1}{z^2}$ and Γ be the straight line joining the points i and $3+i$. Show that 3+3

$$\left|\int_{\Gamma} f(z)\,dz\right| \le 3.$$

(b) Evaluate $\displaystyle\int_{|z|=2} \frac{1}{(z^2+1)}\,dz$.

GROUP-C

Answer any *two* questions from the following 12×2 = 24

8. (a) Prove that union of two compact subsets of the metric space $(X,\ d)$ is also compact. 6

(b) Let $(X,\ d)$ be a metric space with $x_0 \in X$. Let $f: X \longrightarrow \mathbb{R}$ be defined by $f(x)=d(x,\ x_0)$. Prove that f is uniformly continuous on X. 6

9. (a) State and prove Cauchy-Goursat theorem. 8

(b) Find the value of $\displaystyle\int_{\Gamma} \frac{dz}{z-a}$, if 4

(i) a lies inside Γ, and

(ii) a lies outside Γ.

10.(a) If $f(z)$ is analytic within and on a simple closed rectifiable curve Γ and z_0 is any point inside Γ, then prove that $f'(z_0)=\dfrac{1}{2\pi i}\displaystyle\int_{\Gamma} \frac{f(z)}{(z-z_0)^2}\,dz$. 6

(b) Expand $f(z)=\cos z$ in Taylor Series about $z=\pi/4$ and determine the region of convergence of the series. 6

11.(a) Suppose that $f(z)=u(x,\ y)+iv(x,\ y)$ be an entire function such that $u_y - v_x = -2$ for all $z(=x+iy)\in\mathbb{C}$. Verify the function $f(z)$ is constant or not. 5

(b) Prove that every polynomial of degree n has exactly n (not necessarily distinct) zeros. 3

(c) Evaluate $\displaystyle\int_{\Gamma} \frac{\log z}{(z-1)^3}\,dz$, where Γ is the circle $|z-2|=3/2$. 4

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